# 6) Appendix: Stored energy densities

1. Instantaneous Poynting theorem

In time domain:

$$\vec{\mathcal{H}} \cdot \left(\vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \mathcal{B}\right) \qquad \left[\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}\right]$$

$$-\vec{\mathcal{E}} \cdot \left(\vec{\nabla} \times \vec{\mathcal{H}} = \vec{\mathcal{J}} + \frac{\partial \vec{\mathcal{D}}}{\partial t}\right)$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) = \vec{\nabla} \cdot \vec{S} = -\vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{D}} - \vec{\mathcal{H}} \cdot \partial_t \vec{\mathcal{B}} - \vec{\mathcal{J}} \cdot \vec{\mathcal{E}}$$

$$\Rightarrow \quad \underbrace{-\vec{\mathcal{J}} \cdot \vec{\mathcal{E}}}_{\text{power provided by } \vec{\mathcal{J}}} = \underbrace{\vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{D}} + \vec{\mathcal{H}} \cdot \partial_t \vec{\mathcal{B}}}_{?} + \underbrace{\vec{\nabla} \cdot \vec{S}}_{\text{power leaving}}$$

? When non dispersive: $\vec{\mathcal{D}} = \varepsilon_0 \vec{\mathcal{E}}$, $\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}$, it is just $\partial_t \left( \overbrace{\frac{1}{2} \varepsilon_0 \vec{\mathcal{E}}^2}^{W_e(t)} + \overbrace{\frac{1}{2} \mu_0 \vec{\mathcal{H}}^2}^{W_m(t)} \right)$

And the time harmonic averages are

$$\boxed{\begin{aligned} \tilde{W}_e(t) &= \frac{1}{4} \varepsilon_0 |\vec{E}|^2 \\ \tilde{W}_h(t) &= \frac{1}{4} \mu_0 |\vec{H}|^2 \end{aligned}}$$

average stored electric and magnetic energy densities

when $\vec{\mathcal{E}} = |\vec{E}| \cos(\omega_0 t) \qquad \vec{\mathcal{H}} = |\vec{H}| \cos(\omega_0 t)$

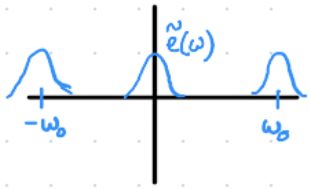
When dispersive $\Rightarrow$ not yet so clear!

2. Time-harmonic field with slowly varying enveloppe

let's consider a quasi monochromatic field

$\vec{\mathcal{E}}(t) = \vec{e}(t)\cos(\omega_0 t)$, with $\vec{e}(t)$ being a slow envelope (we will take the limit $\vec{e}(t)$ $\infty$-ly slow at the end, I need some bandwidth to sense dispersion):

$$\vec{\mathcal{E}}(t) = \vec{e}(t)\cos(\omega_0 t)$$




Convolution of the Fourier spectra: $\vec{E}(\omega) = \frac{1}{2}\left[\tilde{e}(\omega-\omega_0) + \tilde{e}(\omega+\omega_0)\right]$

Calculation of $\frac{\partial \mathcal{D}}{\partial t}$

$$\vec{\mathcal{D}}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \vec{D}(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varepsilon(\omega)\vec{E}(\omega) e^{j\omega t} d\omega =$$

$$\frac{1}{4\pi}\left(\int_{-\infty}^{+\infty} \varepsilon(\omega)\tilde{e}(\omega-\omega_0) e^{j\omega t} d\omega + \int_{-\infty}^{+\infty} \varepsilon(\omega)\tilde{e}(\omega+\omega_0) e^{j\omega t} d\omega\right) =$$

$\Omega = \omega - \omega_0$ $\qquad$ $-\Omega = \omega + \omega_0$

$$\frac{1}{4\pi}\left(\int_{-\infty}^{+\infty} \varepsilon(\Omega+\omega_0)\tilde{e}(\Omega) e^{j(\Omega+\omega_0)t} d\Omega + \int_{\substack{+\infty \\ -\infty}}^{+\infty} \varepsilon(-\Omega-\omega_0)\tilde{e}(-\Omega) e^{-j(\Omega+\omega_0 t)} (+d\Omega)\right)$$

Real signal condition: $\tilde{e}(-\Omega) = \tilde{e}^*(\Omega)$

Crossing condition: $\varepsilon(-\Omega) = \varepsilon^*(\Omega)$ $\qquad$ $z + z^* = 2\,\mathrm{Re}\, z$

$$\Rightarrow \vec{\mathcal{D}}(t) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \varepsilon(\Omega+\omega_0)\tilde{e}(\Omega) e^{j(\Omega+\omega_0)t} d\Omega\right\}$$

$$\Rightarrow \frac{\partial \vec{\mathcal{D}}}{\partial t} = \frac{1}{2\pi}\mathrm{Re}\left\{j e^{j\omega_0 t}\int_{-\infty}^{+\infty} (\Omega+\omega_0)\varepsilon(\Omega+\omega_0)\tilde{e}(\Omega) e^{j\Omega t}\right\}$$

Taylor expansion: We assume $\omega\,\varepsilon(\omega)$ not to change a lot within the bandwidth of the envelope (pick it small enough)

$$(\Omega + \omega_0)\,\varepsilon(\Omega + \omega_0) \simeq \omega_0 \varepsilon(\omega_0) + \Omega \left.\frac{\partial(\omega\varepsilon)}{\partial\omega}\right|_{\omega_0}$$

$$\Rightarrow \frac{\partial \vec{\mathcal{D}}}{\partial t} = \frac{1}{2\pi} \mathrm{Re}\left[ j e^{j\omega_0 t} \int_{-\infty}^{+\infty} \left( \omega_0 \varepsilon(\omega_0) \tilde{e}(\Omega) e^{j\Omega t} d\Omega + \Omega \left.\frac{\partial(\omega\varepsilon)}{\partial\omega}\right|_{\omega_0} \tilde{e}(\Omega) e^{j\omega t} \right) d\Omega \right.$$

$$= \mathrm{Re}\left[ e^{j\omega_0 t} j\, \omega_0 \varepsilon(\omega_0) \underbrace{\left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{e}(\Omega) e^{j\Omega t} d\Omega \right)}_{e(t)} + e^{j\omega_0 t} \left.\frac{\partial(\omega\varepsilon)}{\partial\omega}\right|_{\omega_0} \underbrace{\left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} j\Omega\, \tilde{e}(\Omega) e^{j\omega t} d\Omega \right)}_{\frac{\partial e(t)}{\partial t}} \right]$$

$$\Rightarrow \frac{\partial \vec{\mathcal{D}}}{\partial t} = \mathrm{Re}\left[ e^{j\omega_0 t} j\ \varepsilon(\omega_0) \right] \omega_0 \vec{e}(t) + \mathrm{Re}\left[ e^{j\omega_0 t} \left.\frac{\partial(\omega\varepsilon)}{\partial\omega}\right|_{\omega_0} \right] \frac{\partial \vec{e}(t)}{\partial t}$$

$$\mathrm{Re}\left[ e^{j\omega_0 t} j\, \varepsilon(\omega_0) \right] = \mathrm{Re}\left[ (\cos\omega_0 t + j \sin\omega_0 t)\left( j\varepsilon'(\omega_0) - \varepsilon''(\omega_0) \right) \right]$$

$$= -\varepsilon''(\omega_0)\cos\omega_0 t - \varepsilon'(\omega_0)\sin\omega_0 t$$

$$\mathrm{Re}\left[ e^{j\omega_0 t} \left.\frac{\partial(\omega\varepsilon)}{\partial\omega}\right|_{\omega_0} \right] = \mathrm{Re}\left[ (\cos\omega_0 t + j\sin\omega_0 t)\left( \mathrm{Re} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} + j\, \mathrm{Im} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \right) \right.$$

$$= \mathrm{Re} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \cos\omega_0 t - \mathrm{Im} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \sin\omega_0 t$$

$$\Rightarrow \frac{\partial \vec{\mathcal{D}}(t)}{\partial t} = -\omega_0 \varepsilon'(\omega_0) \sin\omega_0 t\, \vec{e}(t) + \mathrm{Re} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \cos\omega_0 t \frac{\partial \vec{e}}{\partial t} \quad \leftarrow \text{term due to } \varepsilon'$$

$$- \omega_0 \varepsilon''(\omega_0) \cos\omega_0 t\, \vec{e}(t) - \mathrm{Im} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \sin\omega_0 t \frac{\partial \vec{e}}{\partial t} \quad \leftarrow \text{term due to } \varepsilon''$$

use $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$

$$\Rightarrow \overbrace{\vec{\mathcal{E}}(t)}^{\vec{e}(t)\cos\omega_0 t} \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} = -\frac{1}{2}\omega_0 \varepsilon'(\omega_0) \sin(2\omega_0 t) \vec{e}^{\,2}(t) + \mathrm{Re} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \cos^2\omega_0 t\, \overbrace{\vec{e}(t) \cdot \frac{\partial \vec{e}}{\partial t}}^{\frac{1}{2}\frac{\partial \vec{e}^{\,2}}{\partial t}}$$

$$- \omega_0 \varepsilon''(\omega_0) \cos^2(\omega_0 t)\, \vec{e}^{\,2}(t) - \frac{1}{2} \mathrm{Im} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \sin(2\omega_0 t)\, \overbrace{\vec{e}(t) \cdot \frac{\partial \vec{e}}{\partial t}}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{\mathcal{E}}(t) \cdot \frac{\partial \vec{\mathcal{D}}}{\partial t} &= \frac{1}{2} \mathrm{Re} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \cos^2\omega_0 t \frac{\partial \vec{e}^{\,2}}{\partial t} - \omega_0 \varepsilon''(\omega_0)\cos^2(\omega_0 t)\, \vec{e}^{\,2}(t) \\ &\quad - \frac{1}{2}\omega_0 \varepsilon'(\omega_0) \sin(2\omega_0 t)\, \vec{e}^{\,2}(t) - \frac{1}{4} \mathrm{Im} \left.\frac{\partial\omega\varepsilon}{\partial\omega}\right|_{\omega_0} \sin 2\omega_0 t \frac{\partial \vec{e}^{\,2}}{\partial t} \end{aligned}}$$

3. Time-averaged Poynting theorem

We now calculate the average of $\vec{\mathcal{E}}.\frac{\partial}{\partial t}\vec{\mathcal{D}}$ over a period $T_0 = \frac{2\pi}{\omega_0}$ in the limit of infinitely slow envelope $\vec{e}(t) \Rightarrow \vec{e}^2(t)$ and $\frac{\partial \vec{e}^2}{\partial t}$ are quasi constant over $T_0$

$$\frac{1}{T_0}\int_0^{T_0} \vec{\mathcal{E}}\partial_t\vec{\mathcal{D}} = \langle \vec{\mathcal{E}}\partial_t\vec{\mathcal{D}} \rangle = \frac{1}{4}\,\mathrm{Re}\,\frac{\partial \omega\varepsilon}{\partial \omega}\Big|_{\omega_0} \frac{\partial \vec{e}^2}{\partial t^2} - \frac{1}{2}\,\omega_0\,\varepsilon''(\omega_0)\,\vec{e}^2(t) + 0 + 0$$

$$\Rightarrow \langle \vec{\mathcal{E}}.\partial_t\vec{\mathcal{D}} \rangle = \partial_t\left[\frac{1}{4}\,\mathrm{Re}\,\frac{\partial \omega\varepsilon}{\partial \omega}\Big|_{\omega_0} \vec{e}^2(t)\right] - \frac{1}{2}\,\omega_0\,\varepsilon''(\omega_0)\,\vec{e}^2(t)$$

$$\langle \vec{\mathcal{H}}.\partial_t\vec{\mathcal{B}} \rangle = \partial_t\left[\frac{1}{4}\,\mathrm{Re}\,\frac{\partial \omega\mu}{\partial \omega}\Big|_{\omega_0} \vec{h}^2(t)\right] - \frac{1}{2}\,\omega_0\,\mu''(\omega_0)\,\vec{h}^2(t)$$

$$\boxed{\begin{aligned} \tilde{W}_e &= \frac{1}{4}\,\mathrm{Re}\,\frac{\partial \omega\varepsilon}{\partial \omega}\Big|_{\omega_0} \vec{e}^2(t) \\ \tilde{W}_m &= \frac{1}{4}\,\mathrm{Re}\,\frac{\partial \omega\mu}{\partial \omega}\Big|_{\omega_0} \vec{h}^2(t) \end{aligned}}$$

average stored electric and magnetic energies

depend on t just because of $\vec{e}(t)$

Poynting theorem in dispersive media:

$$\Rightarrow \boxed{\underbrace{-\langle \vec{\mathcal{J}}.\vec{\mathcal{E}} \rangle}_{\text{average power delivered by sources}} = \underbrace{\vec{\nabla}.\langle \vec{S} \rangle}_{\text{power flow away}} + \underbrace{\partial_t(\tilde{W}_e + \tilde{W}_m)}_{\text{variation of stored energy}} \underbrace{- \frac{1}{2}\,\omega_0\,\varepsilon''(\omega_0)\langle \vec{\mathcal{E}}^2 \rangle}_{\text{dielectric losses}} \underbrace{- \frac{1}{2}\,\omega_0\,\mu''(\omega_0)\langle \vec{\mathcal{H}}^2 \rangle}_{\text{magnetic losses}}}$$

4. Relation with complex Poynting theorem

Complex phasors at $\omega_0$: $\vec{H}^* \cdot \left(\nabla \times \vec{E} = -j\omega_0 \vec{B}\right)$ (static envelope in this case)

$$\left(\vec{\nabla} \times \vec{H} = \vec{J} + j\omega_0 \vec{D}\right)^* \cdot \vec{E}$$

$$\Rightarrow \vec{H}^* \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{H}^* = \vec{\nabla} \cdot (\vec{E} \times \vec{H}^*) = -j\omega_0 \vec{B} \cdot \vec{H}^* + j\omega_0 \vec{E} \cdot \vec{D}^* - \vec{E} \cdot \vec{J}^*$$

complex poynting vector

$$\Rightarrow \vec{\nabla} \cdot \left(\frac{1}{2} \vec{E} \times \vec{H}^*\right) = \vec{\nabla} \cdot \vec{S} \Rightarrow \boxed{-\frac{1}{2} \vec{E} \cdot \vec{J}^* = \vec{\nabla} \cdot \vec{S} + j\omega_0 \left(\frac{1}{2} \vec{B} \cdot \vec{H}^* - \frac{1}{2} \vec{E} \cdot \vec{D}^*\right)}$$

complex Poynting th.

$$-\frac{1}{2} \vec{E} \cdot \vec{D}^* = -\frac{1}{2} (\varepsilon_R - j\varepsilon_I) |E|^2 \Rightarrow -j\omega_0 \frac{1}{2} \vec{E} \cdot \vec{D}^* = -\frac{1}{2} \omega_0 \varepsilon_I |E|^2 - \frac{j\omega_0}{2} \varepsilon_R |E|^2$$

$$+j\omega_0 \frac{1}{2} \vec{B} \cdot \vec{H}^* = +\frac{1}{2} j\omega_0 (\mu_R + j\mu_I) |\vec{H}|^2 = -\frac{1}{2} \omega_0 \mu_I |\vec{H}|^2 + \frac{j\omega_0}{2} \mu_R |\vec{H}|^2$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\left[-\frac{1}{2} \vec{E} \cdot \vec{J}^*\right] &= \vec{\nabla} \cdot (\mathrm{Re}\, \vec{S}) - \frac{1}{2} \omega_0 \varepsilon_I |\vec{E}|^2 - \frac{1}{2} \omega_0 \mu_I |\vec{H}|^2 \\ j\, \mathrm{Im}\left[-\frac{1}{2} \vec{E} \cdot \vec{J}^*\right] &= j \vec{\nabla} \cdot (\mathrm{Im}\, \vec{S}) + j\omega_0 \left(\frac{1}{2} \mu_R |\vec{H}|^2 - \frac{1}{2} \varepsilon_R |\vec{E}|^2\right) \end{aligned}}$$

Re = time average power balance = consistent with dispersive formula

for $\vec{E} = |\vec{E}| \cos(\omega_0 t)$ $\quad \vec{e}(t) = |\vec{E}|$ $\quad \left(\partial_t (\tilde{W}_e + \tilde{W}_m) = 0\right)$

Im = time average reactive power balance: related to $\tilde{W}_e$ and $\tilde{W}_m$ when $\partial_\omega = 0$:

$$2 \tilde{W}_e = \frac{1}{2} \mathrm{Re} \frac{\partial \omega \varepsilon}{\partial \omega}\bigg|_{\omega_0} |E|^2 = \frac{1}{2} \varepsilon_R |\vec{E}|^2 + \frac{1}{2} \omega_0 \frac{\partial \varepsilon_R}{\partial \omega}\bigg|_{\omega_0} |\vec{E}|^2$$

$$\Rightarrow \boxed{\mathrm{Im}\left[-\frac{1}{2} \vec{E} \cdot \vec{J}^*\right] = \vec{\nabla} \cdot (\mathrm{Re}\, \vec{S}) + 2\omega_0 \left(\tilde{W}_m - \tilde{W}_e + \frac{1}{4} \omega_0 \frac{\partial \varepsilon_R}{\partial \omega}\bigg|_{\omega_0} |\vec{E}|^2 - \frac{1}{4} \omega_0 \frac{\partial \mu_R}{\partial \omega}\bigg|_{\omega_0} |\vec{H}|^2\right)}$$